commerce undergraduate society

# MATH104/184 REVIEW SESSION

BY HANNAH CHUNG

CMP

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# I. Introduction and Few Tips

**Hello!**
I hope you are having a great term so far.

Here are few great tips for your MATH 104/184 Midterm:

1. ***Practice, practice, and practice*.**

2. ***If you don't know how to solve a question, come back to it.***
   The midterm can be very long and the time given to you may be very short. Don't waste your time on one question.

3. ***Make sure you write down all the information that is given to you. (or highlight some important points)***
   Sometimes, when you are nervous, it is easy to make mistakes.

4. ***Watch out for signs***
   Remember, negative and positive signs can result in very different answers.

If you have any question, feel free to ask! I am more than happy to help you out. Good Luck!

## II. Review of Exponentials, Logarithms, and Inverse Functions

Things to Remember:

| | |
|---|---|
| $b^x * b^y = b^{x+y}$ | $y = b^x \leftrightharpoons \log_b y = x$ |
| $b^x / b^y = b^{x-y}$ | $\log_b(xy) = \log_b x + \log_b y$ |
| $(b^x)^y = b^{x*y}$ | $\log\left(\frac{x}{y}\right) = \log x - \log y$ |
| | $\log(x^y) = y\log(x)$ |

**Question 2(a):** Find the inverse of $f(x)=x^3-9$

Solution 2(a):

$y=f(x)= x^3-9$

$x^3=y+9$

$x=\sqrt[3]{y+9}$

$\therefore y=f^{-1}(x)= \sqrt[3]{x+9}$

**Question 2(b):** A culture of bacteria has a population of 150 cells when it is first observed. The population doubles every 12hr. How long does it take the population to reach 10,000?

Solution 2(b):

$P = 150(2)^{\frac{t}{12}}$, *where t = number of hours*

$10{,}000 = 150 * 2^{t/12}$

$\frac{10{,}000}{150} = 2^{\frac{t}{12}}$

$66.\overline{66} = 2^{\frac{t}{12}}$

$\ln(66.\overline{66}) = \ln 2^{\frac{t}{12}}$

$\ln(66.\overline{66}) = \frac{t}{12}\ln 2$

$12\ln(66.\overline{66}) = t\ln 2$

$t = \frac{12\ln(66.\overline{66})}{\ln 2} \approx 72.70$ hours

$P = Ae^{kt}$

$P = 150e^{12k}$

$P = 150e^{12k}$

$300 = 150e^{12k}$

$\ln 2 = \ln e^{12k}$

$\ln 2 = 12k$

$k = \frac{\ln 2}{12}$

$P(t) = 150e^{\left(\frac{\ln 2}{12}\right)t} = 10{,}000$

$e^{kt} = \frac{10{,}000}{150} = \frac{200}{3}$

$kt = \ln(\frac{200}{3})$

$t = \frac{\ln\left(\frac{200}{3}\right)}{k} = \frac{\ln\left(\frac{200}{3}\right)}{\frac{ln2}{12}}$

$t = \frac{12[\ln(200) - \ln(3)]}{ln2}$

## III. A Standard Business Problem

Things to Remember:

| |
|---|
| Cost Function C = C(q), where p= price and q=quantity demanded |
| Revenue Function R=R(q)=p*q |
| Break-even Points: C(q)=R(q) |
| Profit Function P(q)=R(q)-C(q) |

**Question 3:** Jenny decides to start a new cookie business. She estimates that when the price of the cookie is $20, then the daily demand for it is 500 units. For every $1 increase in the price, the daily demand decreases by 5 units. Assume that the fixed costs of production on a daily basis are $10,000, and the variable costs of production are $7 per unit.

(a) Find the linear demand equation for the cookie. Use the notation p for the unit price and q for the daily demand.

Solution 3(a):

$$\text{Slope} = \frac{\Delta p}{\Delta q} = -\frac{1}{5}$$

$$P(q) = aq + b$$

$$P(q) = -\frac{1}{5}q + b$$

$$20 = -\frac{1}{5}(500) + b$$

$$20 + \frac{1}{5}(500) = b = 120$$

$$\therefore P(q) = -\frac{1}{5}q + 120$$

(b) Find the weekly cost function, C=C(q), for producing q cookies per day.

Solution 3(b):

Fixed cost does not change even after you increase the number of units you are producing. Variable Cost, however, changes, as you increase the number of units. The numbers are given in the question: 10,000 fixed costs of production, and variable costs of $7 per unit.

$$\therefore C(q) = 10{,}000 + 7q$$

(c) Find the daily revenue function, R=R(q).

Solution 3(c):

$\mathrm{R(q)} = \mathrm{P(q)} * \mathrm{q}$

$\mathrm{R(q)} = \left(-\frac{1}{5}\mathrm{q} + 120\right)\mathrm{q} = -\frac{1}{5}q^2 + 120q$

$\therefore \mathrm{R(q)} = -\frac{1}{5}q^2 + 120q$

(d) Find the break-even points where Cost equals Revenue.

Solution 3(d):

$\mathrm{R(q)} = \mathrm{C(q)}$

$-\frac{1}{5}\mathrm{q}^2 + 120\mathrm{q} = 10{,}000 + 7\mathrm{q}$

$-\frac{1}{5}\mathrm{q}^2 + 120\mathrm{q} - 7\mathrm{q} - 10{,}000 = 0$

$-\frac{1}{5}\mathrm{q}^2 + 113\mathrm{q} - 10{,}000 = 0$

$\frac{1}{5}\mathrm{q}^2 - 113\mathrm{q} + 10{,}000 = 0$

Use the Quadratic Formula:

$$\mathrm{q} = \frac{-b \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2a}$$

$$\mathrm{q} = \frac{113 \pm \sqrt{(-113)^2 - 4\left(\frac{1}{5}\right)(10{,}000)}}{2\left(\frac{1}{5}\right)} = \frac{113 \pm \sqrt{4769}}{\frac{2}{5}} = \frac{5}{2}\left(113 \pm \sqrt{4769}\right)$$

$= 455.14448667 \; or \; 109.8551333$

$\therefore$ Approximately 455 cookies or 110 cookies

Note that you will not be allowed to use a calculator.

Writing $\mathrm{q} = \frac{113 \pm \sqrt{(-113)^2 - 4\left(\frac{1}{5}\right)(10{,}000)}}{2\left(\frac{1}{5}\right)}$ as your final answer will be enough.

(e) How should Jenny operate in order to maximize the daily profit P=P(q)?

Solution 3(e):

Profit $P(q) = R(q) - C(q)$

Profit $P(q) = -\frac{1}{5}q^2 + 120q - (10{,}000 + 7q)$

Profit $P(q) = -\frac{1}{5}q^2 + 120q - 7q - 10{,}000$

Profit $P(q) = -\frac{1}{5}q^2 + 113q - 10{,}000$

Profit $P(q) = -\frac{1}{5}q^2 + 113q - 10{,}000$

$\frac{dv}{dq}P(q) = P'(q) = -\frac{2}{5}q^{2-1} + 113(1)q^{1-1}$

$P'(q) = -\frac{2}{5}q + 113$

$Set\ P'(q) = 0\ for\ maximum\ Revenue$

$0 = -\frac{2}{5}q + 113$

$\frac{2}{5}q = 113$

$q = 113 * \left(\frac{5}{2}\right) = 282.5$

$\therefore q = 282.5\ cookies$

Alternative Explanation: Since it is a parabola, the maximum will be at the mid-point of the roots. For example, (455+110)/2= 282.5 cookies

## IV. Limits

**Question 4(a):** Evaluate the limit

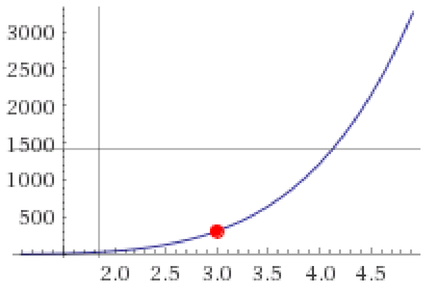
$$\lim_{x \to 3} (x^2 - 2)(x^3 + 6x - 1)$$

Solution 4(a):

$$\lim_{x \to 3} (x^2 - 2)(x^3 + 6x - 1) = (3^2 - 2)(3^3 + 6(3) - 1) = (9 - 2)(27 + 18 - 1) = 308$$

This means that as x approaches 3, the function f(x) approaches 308

(Images from http://www.wolframalpha.com/input/?i=lim+x-%3E3+%5B%28x%5E2-2%29%28x%5E3%2B6x-1%29%5D)




**Question 4(b):** Evaluate the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Solution 4(b):

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} x + 2 \ = 2 + 2 = 4$$

**Question 4(c):** Evaluate the limit

$$\lim_{x \to 0} \frac{(2 + x)^3 - 8}{x}$$

Solution 4(c):

$$\lim_{x \to 0} \frac{(2+x)^3-8}{x} = \lim_{x \to 0} \frac{(2+x)(2+x)^2-8}{x} = \lim_{x \to 0} \frac{(2+x)(4+4x+x^2)-8}{x} = \lim_{x \to 0} \frac{8+8x+2x^2+4x+4x^2+x^3-8}{x} =$$

$$\lim_{x \to 0} \frac{x^3+6x^2+12x+8-8}{x} = \lim_{x \to 0} \frac{x^3+6x^2+12x}{x} = \lim_{x \to 0} x^2 + 6x + 12 = 0^2 + 6(0) + 8 = 12$$

## V. Continuous Functions

Things to Remember:

In order for the function f(x) to be continuous at a, the following conditions must be met:

- **f(a) is defined**
- **$\lim_{x\to a} f(x)$ exists (that is, the right sided and left sided limit must equal)**
- **$\lim_{x\to a} f(x) = f(a)$**

**Question 5(a):** If f and g are continuous functions with f(0)=6 and

$\lim_{x\to 0} f(x)g(x) = 36$, find g(0).

Solution 5(a):

If g and f are continuous at a, $\lim_{x\to 0} f(x)g(x) = f(0)g(0)$

36= f(0)g(0)=6*g(0)

g(0)=6

**Question 5(b):** For what values of c is the function g continuous on (-∞,∞)?

$$g(x) = x^2 \text{ if } x < 2,$$
$$= cx + 20 \text{ if } x \geq 2$$

Solution 5(b):

Remember, $\lim_{x\to 2}$ f(x)=f(2) in order for g(x) to be continuous.

$\lim_{x\to a+} f(x)$=L

$\lim_{x\to a-} f(x)$ =L

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a-} f(x)$=L and $\lim_{x\to a+} f(x)$=L

$\lim_{x\to 2+} (cx + 20)$=L

$\lim_{x\to 2-} x^2$ =L

$\lim_{x\to 2+} (cx + 20)$= $\lim_{x\to 2-} x^2$

$2c + 20 = 2^2$

$2c = -16$

**c=-8**

## VI. Intermediate Value Theorem

Things to Remember:

| |
|---|
| **Intermediate Value Theorem:** f(x) is continuous on interval [a,b] and L is a number strictly between f(a) and f(b). Then there exists at least one number c in (a,b) satisfying f(c)=L. (Or, if f is continuous on [a,b], then the graph of f(x) must cross the horizontal line y=L at least once). |

**Question 6:** Suppose you invest $100 in a 10-year savings account with a fixed annual interest rate r, with yearly compounding. The amount of money A in the account after 10 years is $A(r) = 100(1 + r)^{10}$. You want to have $1000 in the account after 10 years.

(a) Use the Intermediate Value Theorem to show there is an interest rate between 0% and 40% for which A(r)=1000.

Solution 6(a):
Here, we can use Intermediate Value Theorem because A(r) is continuous and we have a closed interval [0, 0.40]. A(r) is a continuous function as it is a shifted exponential function.

$A(r) = 100(1 + r)^{10}$

$A(0) = 100(1 + 0)^{10} = 100$

$A(0.40) = 100(1 + 0.40)^{10} = 2892.55$

$$A(0.40) < 1000 < A(0.40)$$

According to the Intermediate Value Theorem, there is a value of r in (0, 0.40) for which A(r)= 1000.

## VII. Rules of Differentiation

Things to Remember:

**Derivative:** Slopes of the tangent line = instantaneous rate of change in f at a
**The average rate of change** in f on the interval [a,a+h] is the slope of the corresponding secant line:

$$m_{sec} = \frac{[f(a+h) - f(a)]}{h}$$

**The instantaneous rate of change** in f at a:

$$m_{tan} = \lim_{h \to 0} \frac{[f(a+h) - f(a)]}{h}$$

**Question 7(a):** Find an equation of the line tangent to the graph of
$\mathrm{f(x)} = \mathrm{x}^2 + 4\ at\ (1,5)$

Solution 7(a):

Let a=1

$\mathrm{f}(1+\mathrm{h}) = (1+\mathrm{h})^2 + 4 = 1 + 2h + h^2 + 4 = 5 + 2h + h^2$

$\mathrm{f}(1) = (1)^2 + 4 = 5$

$$m_{tan} = \lim_{h \to 0} \frac{[f(1+h) - f(1)]}{h} = \lim_{h \to 0} \frac{[5 + 2h + h^2 - 5]}{h}$$

$$= \lim_{h \to 0} \frac{[2h + h^2]}{h} = \lim_{h \to 0} (2+h) = 2$$

The line has slope of $m_{tan}$=2 and passes through the point (1,5)

Its equation is y-5=2(x-1) => y=2x+3

Note: Only do this if the question asks for 1st principles or by the "definition of the derivative". Otherwise, you can solve this question by doing the following:

f'(x)=2x

m=f'(1)=2

## A. Constant and Power Rules

Things to Remember:

**Constant Rule:** $\frac{d}{dx}(c) = 0$

**Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$

**Constant Multiple Rule:** $\frac{d}{dx}(cf(x)) = cf'(x)$

## B. Sum Rule

Things to Remember:

**Sum Rule:** $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$

**Question 7A&B (a):** Find the values of $\frac{d}{dz}(9z^{10} + 6z^6 - 8z^2 + 7z + 9)$

Solution 7A&B (a):

$10(9)z^{10-1} + 6(6)z^{6-1} - 8(2)z^{2-1} + 7z^{1-1} + 0$

$= 90z^9 + 36z^5 - 16z + 7$

## C. Product and Quotient Rules

Things to Remember:

**Product Rule:** $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

**Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{[f'(x)g(x) - f(x)g\prime(x)]}{g(x)^2}$

**Question 7C (a):** Find the values of $\frac{d}{dx}x^9(\sqrt{x-9})$

Solution 7C(a):

$$x^9(x-9)^{\frac{1}{2}} = 9x^{9-1}(x-9)^{\frac{1}{2}} + x^9\left(\frac{1}{2}\right)(x-9)^{-\frac{1}{2}} = 9x^8(x-9)^{\frac{1}{2}} + \left(\frac{1}{2}\right)x^9(x-9)^{-\frac{1}{2}}$$

$$9x^8\sqrt{x-9} + \frac{x^9}{2\sqrt{x-9}} = \frac{18x^8(x-9) + x^9}{2\sqrt{x-9}} = \frac{18x^9 - 162x^8 + x^9}{2\sqrt{x-9}}$$

$$= \frac{19x^9 - 162x^8}{2\sqrt{x-9}} = \frac{x^8(19x-162)}{2\sqrt{x-9}}$$

**Question 7C (b):** Find the values of $\frac{d}{dx}\frac{(x^2+3x+4)}{x^2-1}$

Solution 7C (b):

$$\frac{(2x+3)(x^2-1) - (x^2+3x+4)(2x)}{[x^2-1]^2}$$

$$= \frac{[(2x^3 + 3x^2 - 2x - 3) - (2x^3 + 6x^2 + 8x)]}{(x^2-1)^2}$$

$$= \frac{[(3x^2 - 2x - 3) - 6x^2 - 8x\,]}{(x^2-1)^2}$$

$$= \frac{[(-3x^2 - 10x - 3)\,]}{(x^2-1)^2}$$

## D. Chain Rule

Things to Remember:

**Chain Rule:** $\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$

**Chain Rule for Powers:** $\frac{d}{dx}(g(x)^n) = n(g(x))^{n-1} * g'(x)$

**Question 7D (a):** Find the values of $\frac{d}{dx}(6x^2 + 7x + 5)^6$

Solution 7D (a):

$\frac{d}{dx}(6x^2 + 7x + 5)^6$

$=6(6x^2 + 7x + 5)^{6-1}(12x + 7)$

$= 6(6x^2 + 7x + 5)^5(12x + 7)$

**Question 7D (b):** Find the values of $\frac{d}{dx}\sqrt{7x^2 + x + 1}$

Solution 7D (b):

$\frac{d}{dx}\sqrt{7x^2 + x + 1} = \frac{d}{dx}(7x^2 + x + 1)^{\frac{1}{2}}$

$= \left(\frac{1}{2}\right)(7x^2 + x + 1)^{-\frac{1}{2}}(14x + 1) = \left(\frac{1}{2}\right)(14x + 1)(7x^2 + x + 1)^{-\frac{1}{2}}$

## VIII. Trigonometric Derivatives

Things to Remember:

$$\frac{d}{dx}(sinx) = \cos(x)$$

$$\frac{d}{dx}(cosx) = -\sin(x)$$

$$\frac{d}{dx}(tanx) = \sec^2(x)$$

$$\frac{d}{dx}(secx) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(cotx) = -\csc^2(x)$$

$$\frac{d}{dx}(\mathrm{cscx}) = -\csc(x)\cot(x)$$

**Question 8(a):** Calculate the derivative of $y=e^{3x}\cos x$.

Solution 8(a):

$y=e^{3x}\cos x$

$$\frac{dy}{dx} = 3e^{3x}\cos x + e^{3x}(-\sin x)$$

$$\frac{dy}{dx} = 3e^{3x}\cos x - e^{3x}\sin x = e^{3x}(3\cos x - \sin x)$$

**Question 8(b):** Differentiate $y=\sin^{10}x$

Solution 8(b):

$y=\sin^{10}x$

$y'=10\sin^9x(\cos x)$

**Question 8(c):** Differentiate $(\tan(x) + 9)^{26}$

Solution 8(c):

$$\frac{d}{dx}(\tan(x) + 9)^{26} = 26(\tan(x) + 9)^{25}(\sec^2\ x)$$

## IX. Derivatives of $e^x$, Logarithms and Exponentials

Things to Remember:

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}\left(e^{u(x)}\right) = e^u * \frac{du}{dx}$$
$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$
$$\frac{d}{dx}(lnx) = \frac{1}{x}$$
$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}$$
$$\frac{d}{dx}(b^x) = b^x lnb$$
$$\frac{d}{dx}(\log_b x) = \frac{1}{xlnb}$$

**Question 9(a):** Find derivative of $y = \frac{\ln x^3}{x^4}$

Solution 9(a):

$y = \frac{\ln(x^3)}{x^4}$ Be aware: $\ln x^2 \neq (\ln x)^2$

$$\frac{dy}{dx} = \frac{\left[\left(\frac{1}{x^3}\right)(3x^2)(x^4) - \ln(x^3)(4x^3)\right]}{(x^4)^2}$$

$$\frac{dy}{dx} = \frac{\left[\frac{3x^6}{x^3} - 4x^3\ln(x^3)\right]}{(x^4)^2}$$

$$\frac{dy}{dx} = \frac{3x^3 - 4x^3\ln(x^3)}{x^8}$$

$$\frac{dy}{dx} = \frac{x^3(3 - 4\ln(x^3))}{x^8} = \frac{3 - 4\ln(x^3)}{x^5}$$

**Question 9(b):** Find derivative of $y = \cos(x)\ln(cos^3 x)$

Solution 9(b):

$y = \cos(x)\ln(\cos^3 x)$

$\frac{dy}{dx} = (-\sin(x))\ln(\cos^3 x) + \cos x\left(\frac{1}{\cos^3 x}\right)(3\cos^2 x)(-\sin(x))$

$\frac{dy}{dx} = (-\sin(x))\ln(\cos^3 x) - 3\sin x$

$\frac{dy}{dx} = (-\sin x)[\ln(\cos^3 x) + 3]$

Note: It will be easier if you simplify $y = \cos(x)\ln(cos^3 x) = 3\cos(x)\ln(cosx)$

**Question 9(c):** Find derivative of $y = x^{x^9}$

Solution 9(c):

$y = (x)^{x^9}$

$\ln y = x^9 \ln x$

$\frac{d}{dx}\ln y = \frac{d}{dx}x^9 \ln x$

$\frac{d}{dx}\ln y = \frac{d}{dx}x^9 \ln x$

$\frac{1}{y}y' = 9x^8 \ln x + x^9\left(\frac{1}{x}\right)$

$y' = y(9x^8 \ln x + x^8) = \mathbf{x^{x^9} x^8 (9\ln x + 1)}$

## X. Rates of Change – Velocity, Acceleration, and Marginal Cost

**Question 10:** The price-demand and cost functions for the production of products are the following:

$$p = 210 - x$$

$$\text{and cost } C(x) = 50000 + 110x,$$

where x= the number of products and can be sold at a price of p dollars per unit and C(x) is the total cost ($) of producing x units.

(a) Find the marginal cost as a function of x.

Solution 10(a):

Marginal cost=MC= $C'(x)=110$

(b) Find the revenue function in terms of x.

Solution 10(b):

Revenue=p*x= $(210 - x)x = 210x - x^2$

(c) Find the marginal revenue function in terms of x.

Solution 10(c):

Marginal Revenue= $R'(x) = 210 - 2x$

(d) Evaluate the marginal revenue function at x=100.

Solution 10(d):

$R'(100) = 210 - 2(100)$

$=10$

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## XI. Implicit Differentiation

**Question 11(a):** Find dy/dx by implicit differentiation

$$\mathrm{x}^2 - 2xy + y^3 = 5$$

Solution 11(a):

$$\mathrm{x}^2 - 2x\big(y(x)\big) + \big(y(x)\big)^3 = 5$$

$$2\mathrm{x} - 2y - 2x\left(\frac{dy}{dx}\right) + 3(y)^2\left(\frac{dy}{dx}\right) = 0$$

$$2\mathrm{x} - 2y = 2x\left(\frac{dy}{dx}\right) - 3(y)^2\left(\frac{dy}{dx}\right)$$

$$2\mathrm{x} - 2y = \left(\frac{dy}{dx}\right)[2x - 3(y)^2]$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{2\mathrm{x} - 2y}{(2x - 3(y^2))} = \frac{2x - 2y}{2x - 3y^2}$$

**Question 11(b):** Find dy/dx by implicit differentiation at (0,49)

$$\sqrt{\mathrm{x} + \mathrm{y}} = 7 + x^2y^2$$

Solution 11(b):

$$\sqrt{\mathrm{x} + \mathrm{y}} = 7 + x^2y^2$$

$$(x + y)^{\frac{1}{2}} = 7 + x^2(y^2)$$

$$\frac{d}{dx}(x + y)^{\frac{1}{2}} = \frac{d}{dx}[7 + x^2(y)^2]$$

$$\frac{1}{2}(x + y)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 0 + 2x(y)^2 + x^2(2y)\left(\frac{dy}{dx}\right)$$

$$\frac{1}{2}(0 + 49)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 0 + 2(0)(49)^2 + (0)^2(2(49))\left(\frac{dy}{dx}\right)$$

$$\frac{1}{2}(0 + 49)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 0$$

$$\frac{1}{2}\left(\frac{1}{7}\right)\left(1 + \frac{dy}{dx}\right) = 0$$

$$\left(1 + \frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -1$$

## XII. Elasticity of Demand

Things to Remember:

Price Elasticity of Demand (Demand Elasticity) tells us how revenue changes as price changes.

**Price Elasticity of Demand**: $\frac{\mathbf{p}}{\mathbf{q}} * \frac{dq}{dp}$

When $|\varepsilon| > 1$, the good is price elastic ; when there is 1% change in price, there is greater than 1% change in quantity demanded

When $|\varepsilon| < 1$, the good is price inelastic; when there is 1% change in price, there is less than 1% change in quantity demanded

When $|\varepsilon| = 1$, the good is unit elastic; when there is 1% change in price, there is also 1% change in quantity demanded (but not always)

**Question 12:** Suppose the demand curve for cookies is given by q=400-10p

(a) Compute the price elasticity of this demand function.

Solution 12(a):

$$q = 400 - 10p$$

$$\frac{dq}{dp} = -10$$

$$\varepsilon = \frac{p}{q(p)} * \frac{dq}{dp}$$

$$\varepsilon = \frac{p}{q(p)}(-10) = \frac{p}{400 - 10p}(-10) = -\frac{10p}{400 - 10p} = -\frac{p}{40 - p} \ or \ \frac{p}{p - 40}$$

(b) What is the price elasticity of demand when price is $20?

Solution 12(b):

$$\varepsilon = \frac{p}{p - 40} = \frac{20}{20 - 40} = \frac{20}{-20} = -1$$

(c) What is the percentage change in the demand if the price is $20 and increases by 2%?

Solution 12(c):

The unit elasticity $\varepsilon = -1$ indicates that the quantity decreases by 1% when price increases by 1%.

$$-1\text{x}0.02 = -0.02$$

The quantity decreases by 2%.

## XIII. Exponential Growth and Compound Interest

Things to Remember:

$$\mathbf{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

P=principal (the amount you borrow or deposit)
r=annual rate of interest
n=number of times the interest is compounded per year
t=number of years
A= amount of money and interest accumulated after n years

Continuous Interest:

$$\mathbf{A = P(e)^{rt}}$$

**Question 13(a):** Today, Ara has $1000 in her bank account. If it is compounded twice a year, and there is 3% annual interest rate, how much will she have in 2 years?

Solution 13(a):

$$A = 1000\left(1+\frac{0.03}{2}\right)^{2*2} = 1000\left(1+\frac{0.03}{2}\right)^{4} = \$1061.36$$

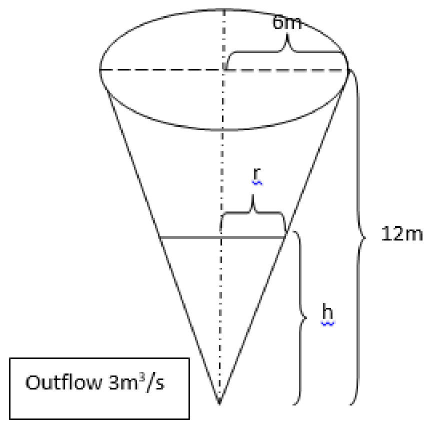
**Question 13(b):** Today, Ara has $1000 in her bank account. A bank is paying an annual interest rate of 3%, compounded continuously. How much will she have in 2 years?

Solution 13(b):

$$A = 1000(e)^{0.03*2} = 1000e^{0.06} = \$1061.84$$

## XIV. Related Rates

**Question 14(a):** An inverted conical water tank with a height of 12m and a radius of 6m is drained through a hole at a rate of $3m^3/s$. What is the rate of change of the water depth when the water depth is 3m?




Solution 14(a):

Let h= the depth of the water in the tank at time

And r= radius of the conical water tank at time t

First, consider a volume function: $V = \frac{\pi}{3} r^2 h$

There are two unknown variables r and h, and we only have one equation:

If you look at the diagram, there are two similar right triangles. (Ratios of h to r will be the same).

By using the similar triangle, $\frac{h}{r} = \frac{12}{6} = 2$

Therefore, $h = 2r$ or $r = \frac{h}{2}$

The volume of the conical water tank:

$$V(t)=\left(\frac{1}{3}\right)\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)(h) = \frac{h^3\pi}{12}$$

$$\frac{d}{dt}V(t) = \frac{d}{dt}\frac{[h(t)]^3\pi}{12}$$

$$V'(t) = \frac{3\pi[h(t)]^2}{12}h'(t) = \frac{\pi h^2}{4}h'(t)$$

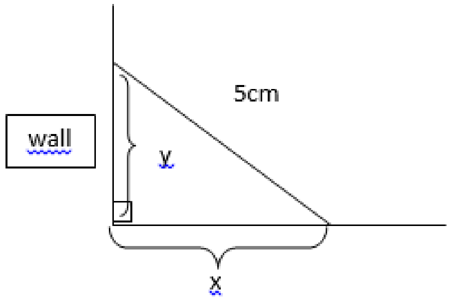
Remember, outflow is the negative rate of change in volume=$-3m^3/s$

We were given that the water depth=h=3m.

$$-3 = \frac{\pi(3)^2}{4}h'(t)$$

$$h'(t) = -\frac{3(4)}{9\pi} = -\frac{\mathbf{12}}{\mathbf{9\pi}}\ \mathbf{m/s} = -\frac{\mathbf{4}}{\mathbf{3\pi}}\mathbf{m/s}$$

**Question 14(b):** You have a 5cm ladder leaning against a wall. If the foot of the ladder slides away from the wall at 1cm/s, how fast is the top of the ladder sliding down the wall when the foot of the ladder is 3 cm away from the wall?




Solution 14(b):
Let x=distance from wall to foot of ladder
y=distance from the floor to the top of the ladder

**Rates in the question:**

$\frac{dy}{dt} < 0, \frac{dx}{dt} > 0$

$\frac{dx}{dt} = +1$ because it is getting further away from the wall

**Use Pythagorean Theorem:**

$x^2 + y^2 = 5^2$

$(x(t))^2 + (y(t))^2 = 5^2$

$\frac{d}{dt}(x(t))^2 + \frac{d}{dt}(y(t))^2 = \frac{d}{dt}5^2$

$2x(t)x'(t) + 2y(t)y'(t) = 0$

$x(t)x'(t) + y(t)y'(t) = 0$

$3(1) + y(t)y'(t) = 0$

**y(t) when x(t)=3**

$5^2 - 3^2 = y^2$

$25 - 9 = 16 = y^2$

$y = 4$

$3(1) + 4y'(t) = 0$

$4y'(t) = -3$

$y'(t) = -\frac{3}{4}$

The ladder is sliding down the wall at -3/4 cm/min.

## XV. Maxima and Minima

Things to Remember:

| |
|---|
| Critical Points: f'(x)=0, or f'(x) does not exist but f(x) does<br>To maximize or minimize f(x) on [a,b] evaluate f(x) at x=a, x=b, and each critical number. |

**Question 15:** Find the absolute maximum and absolute minimum values of

$$f(x) = x^3 - 18x^2 + 108x + 10$$

on the interval [0,10].

Solution 15:

$f(x) = x^3 - 18x^2 + 108x + 10$

$f'(x) = 3x^2 - 36x + 108$

$0 = 3(x^2 - 12x + 36)$

$0 = 3(x - 6)^2$

$x - 6 = 0$

$x = 3, 5$

| $x = 0$ | $x = 6$ | $x = 10$ |
|---|---|---|
| $f(0) = (0)^3 - 18(0)^2 + 108(0) + 10 = 10$ | f(6)=226 | f(10)=290 |

Absolute Maximum on the interval: 290

Absolute Minimum on the interval: 10