MATH 104/184

MIDTERM REVIEW SESSION

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*** means a question is high difficulty level

CMP

I. Function Review

Exponential Function:

$$b^m \times b^n = b^{m+n}$$

$$b^m \div b^n = b^{m-n}$$

$$(b^m)^n = b^{mn}$$

$$b^1 = b$$

$$b^0 = 1$$

Logarithmic Function:

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b(x^n) = n\log_b x$$

$$\log_b(b) = 1$$

$$\log_b(1) = 0$$

1. a) Solve $\log_6 x^7 = 5$ for x.

$7\log_6 x = 5$

$\log_6 x = 5/7$ using log laws

$x = 6^{(5/7)}$ switch from log form to exp form

b) Your student loans of $10,000 have an annual interest rate of 12% compounded continuously. How long will it take for your debt to double?

Double of 10000 = 20000

$20000 = 10000e^{0.12t}$ now solve for t

$2 = e^{0.12t}$

$\ln 2 = \ln (e^{0.12t})$ ln both sides

$\ln 2 = 0.12t \ln (e)$ using log laws

$\ln 2 = 0.12t$ using log laws

$t = \ln 2 / 0.12$

c) Your student loans of $5,000 grew to $12,000 after 30 months. What is the effective interest rate if it was compounded annually?

$FV = PV\ (1 + \frac{i}{n})^{nt}$

$12000 = 5000(1 + i)^{30/12}$

$12/5 = (1 + i)^{5/2}$

$\mathbf{1 + I = (12/5)^{2/5}}$

II. The "Standard Business Problem"

Remember:
p = price and q = quantity
Demand equation = p(q) slope * q + intercept
Cost Function = C(q) = Variable cost * quantity + fixed cost
Revenue Function = R(q) = p*q
Break-even Point is when R(q) = C(q)
Profit Function P(q) = R(q) – C(q)

2. A penguin decides to start a popsicle business. He estimates that when the price of a box of popsicles is $50, the monthly demand will be 300 boxes and for every $2 increase in the price, the monthly demand will fall by 10 boxes. There is also a fixed production cost of $1000 per month and the variable cost are $5 per box.

   a) Find the linear demand equation for boxes of popsicles.

Slope = $\frac{change\ in\ price}{change\ in\ demand} = -\frac{2}{10}$ = -0.2

p = -0.2q + b

50 = -0.2(300) + b

50 = -60 + b

b = 110

p = -0.2q + 110

   b) Find the cost function.

Cost = Variable cost * quantity + fixed cost

c = 5q + 1000

c) Find the monthly revenue function.

Revenue = price * quantity

$R(q) = (-0.2q + 110)q$

$\mathbf{R(q) = -0.2q^2 + 110q}$

d) Find the break-even point(s), leave your answer in a calculator ready form

$R(q) = C(q)$

$-0.2q^2 + 110q = 5q + 1000$

$0 = 0.2q^2 - 105q + 1000$

$$q_1 = \frac{105 + \sqrt{(-105)^2 - 4(0.2)(1000)}}{2(0.2)}$$

$$q_2 = \frac{105 - \sqrt{(-105)^2 - 4(0.2)(1000)}}{2(0.2)}$$

e) Find the optimal price to maximize profit.

$P(q) = R(q) – C(q)$

$P(q) = -0.2q^2 + 105q – (5q + 1000)$

$P(q) = -0.2q^2 + 105q – 1000$

$P'(q) = -0.4q + 105$

$0 = -0.4q + 105$

$0.4q = 105$

$q = 105/0.4 = 105 * (5/2) = 262.5$

then

$p = -0.2(262.5) + 110$ Multiply by 2 then move the decimal place and make it negative then add 110

p = 57.5

f) What is his profit? Leave your answer in calculator ready form

Plug in 262.5 as q for the profit function.

$P(262.5) = -0.2(262.5)^2 + 105(262.5) - 1000$

III. Limits

3. a) Evaluate the limit $\lim_{x\to 7} \frac{x^2-6x-7}{x^2-5x-14}$

$= \lim_{x\to 7} \frac{(x-7)(x+1)}{(x-7)(x+2)}$

$= \lim_{x\to 7} \frac{(x+1)}{(x+2)}$

$= \frac{7+1}{7+2}$

$= \mathbf{\frac{8}{9}}$

b) Evaluate the limit $\lim_{x\to\infty} \frac{8x^9+4x^3+x^2+21}{2x^9+21x^2-4}$

$= \lim_{x\to\infty} \frac{8x^9+4x^3+x^2+21}{2x^9+21x^2-4} * \frac{\frac{1}{x^9}}{\frac{1}{x^9}}$

$= \lim_{x\to\infty} \frac{\frac{8x^9}{x^9}+\frac{4x^3}{x^9}+\frac{x^2}{x^9}+\frac{21}{x^9}}{\frac{2x^9}{x^9}+\frac{21x^2}{x^9}-\frac{4}{x^9}}$ multiplied top and bottom by 1/(highest degree in denominator)

$= \lim_{x\to\infty} \frac{8+\frac{4}{x^6}+\frac{1}{x^7}+\frac{21}{x^9}}{2+\frac{21}{x^7}-\frac{4}{x^9}}$

$= \frac{8+\frac{4}{\infty^6}+\frac{1}{\infty^7}+\frac{21}{\infty^9}}{2+\frac{21}{\infty^7}-\frac{4}{\infty^9}}$ anything divided by infinity is 0

$= \frac{8}{2}$

$= \mathbf{4}$

c) Evaluate the limit $\lim_{x\to\infty} \frac{8x^9+4x^3+x^2+21}{2x^8+21x^2-4}$

$= \lim_{x\to\infty} \frac{8x^9+4x^3+x^2+21}{2x^8+21x^2-4} * \frac{\frac{1}{x^8}}{\frac{1}{x^8}}$ highest degree in denominator this time is x^8

$= \lim_{x\to\infty} \frac{\frac{8x^9}{x^8}+\frac{4x^3}{x^8}+\frac{x^2}{x^8}+\frac{21}{x^8}}{\frac{2x^8}{x^8}+\frac{21x^2}{x^8}-\frac{4}{x^8}}$ multiplied top & bottom by 1/(highest degree in denominator)

$= \lim_{x\to\infty} \frac{8x+\frac{4}{x^5}+\frac{1}{x^6}+\frac{21}{x^8}}{2+\frac{21}{x^6}-\frac{4}{x^8}}$

$= \frac{8\infty+\frac{4}{\infty^5}+\frac{1}{\infty^6}+\frac{21}{\infty^8}}{2+\frac{21}{\infty^6}-\frac{4}{\infty^8}}$ anything divided by infinity is 0

$= \frac{8\infty}{2}$ When the highest degree in the numerator > denominator we are left with infinity

$= \infty$

d) Evaluate the limit $\lim_{x\to\infty} \frac{8x^8+4x^3+x^2+21}{2x^9+21x^2-4}$

$= \lim_{x\to\infty} \frac{8x^8+4x^3+x^2+21}{2x^9+21x^2-4} * \frac{\frac{1}{x^9}}{\frac{1}{x^9}}$ highest degree in denominator is x^9, while we have x^8 in the numerator

$= \lim_{x\to\infty} \frac{\frac{8x^8}{x^9}+\frac{4x^3}{x^9}+\frac{x^2}{x^9}+\frac{21}{x^9}}{\frac{2x^9}{x^9}+\frac{21x^2}{x^9}-\frac{4}{x^9}}$ multiplied top and bottom by 1/(highest degree in denominator)

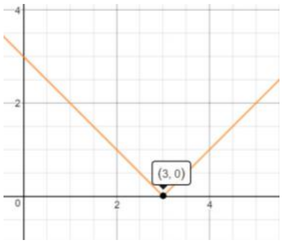
$= \lim_{x\to\infty} \frac{\frac{8}{x}+\frac{4}{x^6}+\frac{1}{x^7}+\frac{21}{x^9}}{2+\frac{21}{x^7}-\frac{4}{x^9}}$

$= \frac{\frac{8}{\infty}+\frac{4}{\infty^6}+\frac{1}{\infty^7}+\frac{21}{\infty^9}}{2+\frac{21}{\infty^7}-\frac{4}{\infty^9}}$ anything divided by infinity is 0

$= \frac{0}{2}$ When the highest degree in the numerator < denominator we are left with 0

$=$ **0**

c) Evaluate the limit $\lim_{x \to 3^-} \frac{|x-3|}{x-3}$



|x-3| looks like this and we can treat it like a piecewise function

|x-3| = $\begin{cases} x-3 & if\ x \geq 3 \\ -(x-3) & if\ x < 3 \end{cases}$

Now we are approaching from 3^- which is LESS than 3. Therefore, we will use –(x -3).

Then the limit is now = $\lim_{x \to 3^-} \frac{-(x-3)}{x-3}$ = **-1**

d) Let f(x) = $\begin{cases} \sin \sqrt[3]{e} & if\ x = e^{\pi} \\ e^{\pi} & if\ x \neq e^{\pi} \end{cases}$, evaluate the limit $\lim_{x \to e^{\pi}} f(f(x))$.

Give your answer in a calculator ready form

Start with the inside f(x) first then work outwards $\lim_{x \to e^{\pi}} f(\boldsymbol{f}(\boldsymbol{x}))$.

As x APPROACHES e^{π}, which also means x is NOT e^{π}, then value of f(x) will be e^{π} (bottom case)

Now we are left with f(e^{π}) which will give us a value of $\sin\sqrt[3]{e}$

Therefore, the final answer is $\mathbf{\sin\sqrt[3]{e}}$

e) Evaluate the limit $\lim_{x \to 16} \frac{\sqrt{x}-4}{16-x}$

= $\lim_{x \to 16} \frac{\sqrt{x}-4}{16-x} * \frac{\sqrt{x}+4}{\sqrt{x}+4}$

= $\lim_{x \to 16} \frac{x-16}{16-x(\sqrt{x}+4)}$ (a+b)*(a-b) = a^2 - b^2

= $\lim_{x \to 16} \frac{-(16-x)}{16-x(\sqrt{x}+4)}$ factor (-1) out from numerator

= $\frac{-1}{(\sqrt{16}+4)}$

= $\frac{-1}{(4+4)}$ = **- $\frac{1}{8}$**

f)*** Evaluate the limit $\lim_{x\to0}\frac{\sqrt[3]{x+1}-1}{x}$ hint use the identity (a-b)(a^2 +ab + b^2) = a^3 – b^3

$= \lim_{x\to0}\frac{\sqrt[3]{x+1}-1}{x} * \frac{\sqrt[3]{x+1}^2+\sqrt[3]{x+1}+1}{\sqrt[3]{x+1}^2+\sqrt[3]{x+1}+1}$ Use the identity to remove the cubic root

$= \lim_{x\to0}\frac{(x+1)-1}{x(\sqrt[3]{x+1}^2+\sqrt[3]{x+1}+1)}$

$= \lim_{x\to0}\frac{x}{x(\sqrt[3]{x+1}^2+\sqrt[3]{x+1}+1)}$

$= \lim_{x\to0}\frac{1}{(\sqrt[3]{x+1}^2+\sqrt[3]{x+1}+1)}$ x in the denominator has been removed we can start plugging in 0 for x

$= \frac{1}{(\sqrt[3]{0+1}^2+\sqrt[3]{0+1}+1)}$

$= \frac{1}{(1+1+1)}$ the whole thing actually turns into a nice fraction

$= \mathbf{\frac{1}{3}}$

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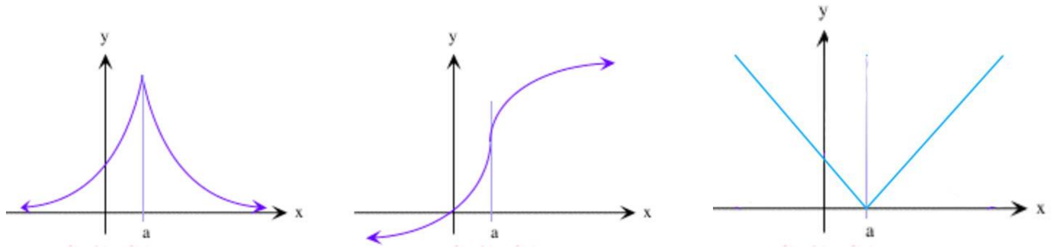
IV. Continuous Functions

A function f(x) is continuous at point a if these three conditions are met:

1. f(a) is defined
2. $\lim_{x \to a} f(x)$ exists and the right and left sided limits are equal
3. $\lim_{x \to a} f(x)$ = f(a)

4. a) True or False:
If g(x) is not differentiable at some point a, then g(x) is not continuous at a. If false draw a graph as a counter argument.

If a function is differentiable it means that we are able to determine the slope. Here are some examples. The 1st and 3rd graph, you cannot draw a tangent line while the 2nd graph has an infinity (undefined) slope at point a.



b) For what value of a will the function f(x) continuous on all x.

$$f(x) = \begin{cases} x^3 + a & if\ x \le \pi \\ 3a\left(\sin\frac{x}{2}\right) & if\ x > \pi \end{cases}$$

f(π) = $\lim_{x \to \pi^-} f(x) = \pi^3 + a$ must be equal to $\lim_{x \to \pi^+} f(x) = 3a(\sin\frac{\pi}{2})$

We can solve $\pi^3 + a = 3a(\sin\frac{\pi}{2})$

$\pi^3 + a = 3a$ $\qquad\qquad$ $\left(\sin\frac{\pi}{2}\right) = 1$

$\pi^3 = 2a$

a = $\frac{\pi^3}{2}$

c) *** For what values of a and b will the function f(x) be continuous for all x

$$f(x) = \begin{cases} 2a + b^x & if\ x \le 0 \\ ax^3 + b(x+1) & if\ 0 < x \le 1 \\ 6a\sqrt[3]{x} & if\ x > 1 \end{cases}$$

Similar concept to previous questions but now with a 3 part piecewise function so we have 2 points to connect.

At x = 0, using part 1 and 2.

$2a + b^0 = a0^3 + b(0+1)$

2a + 1 = b

At x = 1, using part 2 and 3.

$a(1^3) + b(1+1) = 6a\sqrt[3]{1}$

a + 2b = 6a

2b = 5a

Now we just solve the system of equations for a and b.

2(2a+1) = 5a	2b = 5(2)
4a + 2 = 5a	2b = 10
a = 2	**b = 5**

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V. Intermediate Value Theorem

f(x) is continuous on interval [a,b] and L is a number strictly between f(a) and f(b). Then there exist at least one number c in (a,b) that satisfies f(c) = L.

5. Prove using IVT that the equation f(x) = $x^3 - 4x + 1$ has at least 1 zero.

First you should state that because this is a polynomial function that it is continuous everywhere and therefore the IVT applies. Now we need to find an a and b that will give us a sign change.

$f(0) = 1 > 0$

$f(1) = 1 - 4 + 1 = -4 < 0$

Therefore, the IVT guarantees that there is at least 1 value c between 0 and 1 such that f(c) = 0

VI. Definition of Derivative

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ OR}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

6. a) Using the definition of derivative to find the tangent line of f(x) = x^2 at x = 2

Remember $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

= $\lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$

= $\lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h}$

= $\lim_{h \to 0} \frac{2xh+h^2}{h}$ we can now cancel out the h in the denominator

= $\lim_{h \to 0} 2x + h$

= 2x + 0

= 2x This is the derivative (slope)

= 4 at x =2

Now

y = mx + b use f(2) = 4 to get the points (2,4) to plug in

4 = 4(2) + b

b = -4

Therefore, the equation of the tangent line at x =2 is y=4x -4 b)*** Evaluate the limit $\lim_{x\to\pi} \frac{\cos(2x)-1}{x-\pi}$ (hint: recognize the definition of derivative of some function f(x) and derivative it)

You must be able to recognize the definition of derivative with the form $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Notice that the limit approaches π and the denominator is x-π, therefore a = π and f(π) = 1.

Then cos(2x) must be f(x). Therefore, we are just solving for f'(x) using power rule.

f '(x) = − sin(2x) ∗ 2

c)****(SUPER HARD BONUS) Let k(x) = f(x)/ g(x) and using limits show that k'(x) = $\frac{[f'(x) * g(x) - g'(x) * f(x)]}{g(x)^2}$. Hint f '(x) =$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and g'(x) =$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

$$\lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} * \frac{g(x)}{g(x)} - \frac{f(x)}{g(x)} * \frac{g(x+h)}{g(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)}{g(x+h)*g(x)} - \frac{f(x)*g(x+h)}{g(x)*g(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)-f(x)*g(x+h)}{g(x+h)*g(x)}}{h} * \frac{\frac{1}{h}}{\frac{1}{h}} = \lim_{h\to 0} \frac{f(x+h)*g(x)-f(x)*g(x+h)}{h*g(x+h)*g(x)} * \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)}{h} - \frac{f(x)*g(x+h)}{h}}{g(x+h)*g(x)}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)}{h} - \frac{f(x)*g(x+h)}{h} + \frac{f(x)*g(x)}{h} - \frac{f(x)*g(x)}{h}}{g(x+h)*g(x)} = \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)}{h} - \frac{f(x)*g(x)}{h} + \frac{f(x)*g(x)}{h} - \frac{f(x)*g(x+h)}{h}}{g(x+h)*g(x)}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)*g(x)-f(x)*g(x)}{h} + \frac{f(x)*g(x)-f(x)*g(x+h)}{h}}{g(x+h)*g(x)} = \lim_{h\to 0} \frac{g(x)*\frac{f(x+h)-f(x)}{h} + (-1)(f(x))\frac{g(x+h)-g(x)}{h}}{g(x+h)*g(x)}$$

$$= \lim_{h\to 0} \frac{g(x)*f'(x)-f(x)*g'(x)}{g(x+h)*g(x)} = \frac{f'(x)*g(x)-g'(x)*f(x)}{g(x)*g(x)} = \frac{\boldsymbol{f'(x)*g(x)-g'(x)*f(x)}}{\boldsymbol{g(x)^2}}$$

VII. Derivatives

- **Constant Rule:** $f(x) = c$ then $f'(x) = 0$
- **Constant Multiple Rule:** $g(x) = c \cdot f(x)$ then $g'(x) = c \cdot f'(x)$
- **Power Rule:** $f(x) = x^n$ then $f'(x) = nx^{n-1}$
- **Sum and Difference Rule:** $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$
- **Product Rule:** $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
- **Chain Rule:** $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$

f(x)	f ’(x)
sin x	cos x
cos x	- sin x
tan x	$\sec^2 x$
b^x	$b^x \ln b$
e^x	e^x
$\log_b x$	1/(x ln b)
ln x	1/x

7. a) Find the derivative of h(x) = $x^3\sqrt{x^2+1}$. Do not simplify

Product rule. f(x) = x^3 and g(x) = $\sqrt{x^2+1}$

f’(x) = $3x^2$ g’(x) = $\left(\frac{1}{2\sqrt{x^2+1}}\right) * 2x$

(Chain rule so we have to remember to multiply by the derivative of $x^2 + 1$)

h ’(x) = $3x^2\left(\sqrt{x^2+1}\right) + x^3\left(\frac{1}{2\sqrt{x^2+1}}\right) * 2x$

b) Find the value of $(e^{x^3+x-1})'$

$= e^{x^3+x-1} * (x^3 + x - 1)'$

$= e^{x^3+x-1} * (3x^2 + 1)$

c) Find the derivative of cos(sin(2x+2))

Looks ugly but we just need to use chain rule and follow trig derivatives.

Derivative of cos x is - sin x and in this case x = (sin(2x+2) and then use chain rule.

We then derive (sin(2x+2) and multiply by it, but we also have to remember to chain rule again with (2x+2)

$= -\sin((\sin(2x+2)) * \cos(2x+2) * 2$

d) Find the derivative of $\ln((z^3 + 2)^2)$

Another use of chain rule multiple times. First take derivative of ln x where x = $(z^3 + 2)^2$ then we must follow up and multiply by the derivative of $(z^3 + 2)^2$ and then follow up again by multiplying by the derivative of $z^3 + 2$.

$= \frac{1}{(z^3+2)^2} * 2(z^3 + 2) * 3z^2$

VIII. Application of Derivatives

8. a) find the acceleration of an object whose position is s(t) = t sin(2t)

Derivative of position is velocity and the derivative of velocity is acceleration.

Fun fact. The derivative of acceleration is called jerk and then the next 3 derivatives after that are called snap, crackle, and pop. (Rice Krispies is my favourite cereal)

v(t) = sin(2t) + t(cos(2t))(2)   derivative of sin(2t) = cost(2t)*2

a(t) = 2cos(2t) + 2(cost(2t) + t(-sin(2t)2))

= 2cost(2t) + 2cost(2t) + 2t(-sin(2t))

= 4cos(2t) - 4tsin(2t)

b) A snowball is thrown, from a starting height of 0 meters, upwards with an initial velocity of 15m/s and its path is parabolic arc given by the function of time h(t) = 12t – 3t^2. What is the snowball's velocity when it hits its target at h=0?

The velocity of the snowball can be found by taking the derivative of its position function.

v(t) = 12 – 6t

We need to find out the t when the snowball hits.

0 = 12t – 3t^2

0 =3t(4-t)

t= 0 and t=4.

v(4) = 12-6(4)

= -12m/s

IX. Implicit Differentiation

9. a) Find dy/dx using implicit differentiation on $x^2 - 3xy + 2y^3 = 5$

$2x - (3y + 3x\frac{dy}{dx}) + 6y^2\frac{dy}{dx} = 0$

$2x - 3y - 3x\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$

$-3x\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 3y - 2x$

$\frac{dy}{dx}(-3x + 6y^2) = 3y - 2x$

$\frac{dy}{dx} = \frac{3y - 2x}{-3x + 6y^2}$

b) Show that the dy/dx of $e^y = x$ is equal to 1/x using implicit differentiation

$e^y = x$

$e^y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$\frac{dy}{dx} = \frac{1}{x}$ sub in first line to get x back into the equation

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