commerce undergraduate society

MATH 104/184

MIDTERM REVIEW SESSION

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CMP

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Things you should know.

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

θ	0 0	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	U
$\csc(\theta)$	U	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec(\theta)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	U
$\cot(\theta)$	U	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b(x^n) = n\log_b x$$

$$\log_b(b) = 1$$

$$\log_b(1) = 0$$

I. Related Rates

- Application of implicit differentiation
- Taking derivative with respect to time
- Draw a diagram and label which variables are constant and which are changing
- Find an equation that ties the variables together
- Find the rate of change that is needed
- Remember to use the correct units

1. A conical pool has a radius of 3m and a height of 6. The pool drain is opened at the bottom and it is losing water at 2 cubic meter per minute. How fast is the water level dropping when the water is exactly 4m deep?

Want: dh/dt when h = 4m

Given: dV/dt = -2m^3/min

V = $\frac{\pi}{3}r^2h$, diagrams are all on final page.

The problem here is that we have 2 variables in our equation.

However, we can use rewrite one of the variables using the other.

We want to find dh/dt therefore we need to keep h and rewrite r using h.

Notice that 2r = h. Then r=h/2.

V = $\frac{\pi}{3}(\frac{h}{2})^2h$

V = $\frac{\pi}{3}\frac{h^2}{4}h$

V = $\frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{3\pi}{12}h^2\frac{dh}{dt}$

-2 = $\frac{\pi}{4}h^2\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{8}{\pi h^2}$

$\frac{dh}{dt} = -\frac{8}{\pi 4^2} = -\frac{8}{16\pi} = -\frac{1}{2\pi}$ m per min

2. A conical pool has a radius of 3m and a height of 6. The pool drain is opened at the bottom and it is losing water at 2 cubic meter per minute. How fast is the radius shrinking by when the water is exactly 4m deep?

Want: dr/dt when h = 4m

Given: dV/dt = -2m³/min

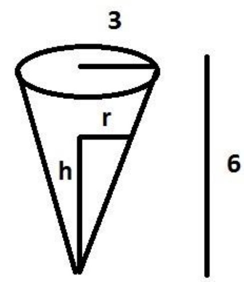
V = $\frac{\pi}{3}r^2h$, btw I made that drawing in Paint because I suck with Photoshop.

The problem here is that we have 2 variables in our equation.

However, we can use rewrite one of the variables using the other.

We want to find dr/dt therefore we need to keep r and rewrite h using r.

Notice that 2r = h.



V = $\frac{\pi}{3}r^2(2r)$

V = $\frac{2\pi}{3}r^3$

$\frac{dV}{dt} = \frac{6\pi}{3}r^2\frac{dr}{dt}$

-2 = $2\pi r^2\frac{dr}{dt}$

$\frac{dr}{dt} = -\frac{1}{\pi r^2}$ 2r = h, h is 4m therefore r is 2m.

$\frac{dr}{dt} = -\frac{1}{\pi 2^2} = -\frac{1}{4\pi} = -\frac{1}{4\pi}$ m per min

OR you could just use this shortcut. 2r = h is true and always true. Therefore, if we were to derive it we get: $2\frac{dr}{dt} = \frac{dh}{dt}$

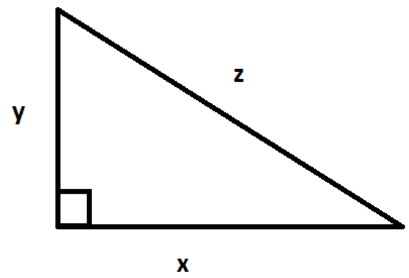
So if we know from question 1 that $\frac{dh}{dt} = -\frac{1}{2\pi}$ we can plug that in to get $2\frac{dr}{dt} = -\frac{1}{2\pi}$ then move then divide both sides by 2 to get $\frac{dr}{dt} = -\frac{1}{4\pi}$ which is the same answer. Cool stuff?

3. A kite is flying at an angle of elevation of π/3. The kite string is being pulled in at a rate of 1 meter per second. If the angle of elevation does not change, how fast is the kite losing altitude?

Want: dy/dt, constant therefore we don't need to worry about at what moment

Given: dz/dt = -1m/s

The equation of this triangle is: $sin\frac{\pi}{3} = \frac{y}{z}$



$\frac{\sqrt{3}}{2} = \frac{y}{z}$

$y = \frac{\sqrt{3}}{2}z$

$\frac{dy}{dt} = \frac{\sqrt{3}}{2}\frac{dz}{dt}$

$\frac{dy}{dt} = \frac{\sqrt{3}}{2}(-1)$

$\frac{dy}{dt} = -\frac{\sqrt{3}}{2}m/sec$

4. A movie is being filmed, the camera is located 40m from a straight highway and a car is traveling on it at 30m/s. The camera turns so it is always pointed at the car. How fast is the camera's angle changing when the car is 40m away from the camera?

Want: $\frac{d\theta}{dt}$ at $\theta = 0$

Given $\frac{dx}{dt} = -30m/s$

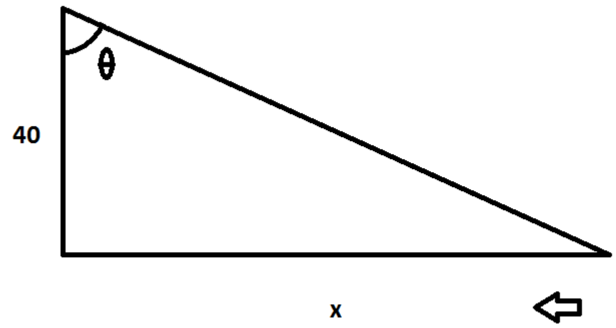
$\tan\theta = \frac{x}{50}$

$sec^2\theta \frac{d\theta}{dt} = \frac{1}{50}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{\frac{1}{50}(-30)}{sec^2\theta}$

$\frac{d\theta}{dt} = \frac{\frac{-3}{5}}{sec^2\theta}$

$\frac{d\theta}{dt} = \frac{-3}{5}$ rad/s



II. Applications of Calculus

1. Differentiate y = x^x, state your answer using only x

lny = ln(x^x)

lny = $xlnx$

$\frac{1}{y}\frac{dy}{dx}$=$x\frac{1}{x} + 1lnx$

$\frac{dy}{dx}$=$(1 + lnx)y$ because the questions says to only have x, we must sub out y

$\frac{dy}{dx}$=$(1 + lnx)\ x^x$ we can use the original equation to sub in for y

2. Differentiate $y = \frac{(x^2+1)^2(x^3+2)^4}{(x+3)^8}$, you do not need to simplify your answer

Don't fall for the trap and start using product and quotient rule. It will turn ugly and so will your mark. Use logarithmic differentiation!

$\ln y = \ln(\frac{(x^2+1)^2(x^3+2)^4}{(x+3)^8})$

$\ln y = \ln(x^2+1)^2 + ln(x^3+2)^4 - ln\,(x+3)^8$

$\frac{dy}{dx}\frac{1}{y} = \frac{1}{(x^2+1)^2}2(x^2+1)(x) + \frac{1}{(x^3+2)^4}4(x^3+2)^3(3x^2) - \frac{1}{(x+3)^8}8(x+3)^7$ remember to chain rule!

$\frac{dy}{dx} = (\frac{1}{(x^2+1)^2}2(x^2+1)(x) + \frac{1}{(x^3+2)^4}4(x^3+2)^3(3x^2) - \frac{1}{(x+3)^8}8(x+3)^7)y$ then just move the y over

III. PRICE ELASTICITY OF DEMAND

$E = \frac{\%\Delta Q}{\%\Delta P} = \frac{dQ/Q}{dP/P} = \frac{P}{Q}\frac{dQ}{dP}$ This is where the formula comes from.

When |E|>1 we say it is price elastic

When |E|<1 we say it is price inelastic

When |E|=1 we say the price is unit elastic, this is the optimal solution that maximizes revenue

1. Air Canada has determined that the demand for its one way tickets from LAX to YVR is: $lnq - lnp + 0.01 = 21$. q is the number of tickets and p is the price in dollars per ticket. What price should they set it at to maximize revenue?

E= $\frac{dq}{dp}\frac{p}{q}$, we need to find $\frac{dq}{dp}$ so we derive $lnq - lnp + 0.01 = 21$ to get:

$\frac{1}{q}\frac{dq}{dp} - \frac{1}{p} + 0.01 = 0$

$\frac{dq}{dp} = \left(\frac{1}{p} - 0.01\right)q$

Then we can plug it in to solve for E.

$E = \left(\frac{1}{p} - 0.01\right)q\frac{p}{q}$

$E = \left(\frac{1}{p} - 0.01\right)p$

E = 1-0.01p then we can set E = -1 because that is where the optimal solution is

-1 = 1-0.01p

-2 = -0.01p

p = 200

Optimal price = $200

2. Now find another way to do question 1

We can find where the marginal revenue = 0 to find the optimal solution as well. When marginal revenue = 0 that means if we were to increase our price by 1 more dollar our revenue would increase by 0 which means we are at the best we can do.

R(p) = pq   (notice it is a function of p because we want to find p)

$\frac{dR}{dp} = q + p\frac{dq}{dp}$ Now we already found $\frac{dq}{dp}$ in question 1 so lets copy that over.

$\frac{dR}{dp} = q + p\left(\frac{1}{p} - 0.01\right)q$

$\frac{dR}{dp} = q + (1 - 0.01p)q$

$\frac{dR}{dp} = q + q - 0.01pq$

$\frac{dR}{dp} = 2q - 0.01pq$ Set marginal revenue to 0

$0 = 2q - 0.01pq$

$0.01pq = 2q$

$0.01p = 2$

$p = 200$

Again, the optimal price = $200

IV. First and Second Derivative Tests

For questions asking for ABSOLUTE max or min.

1.Take derivative

2. Find points where the derivative is equal to 0 or undefined
3. Plug and chug each of those points AND the boundary points for the **y value**
4. Pick out the max and min

For questions asking for LOCAL max or min

1.Take derivative

2. Find points where the derivative is equal to 0 or undefined
3. Draw number line and find the **SIGN** of the **first derivative** of the intervals.

    Negative means decreasing, positive means increasing.

    if it goes from increasing to decreasing it is a local max.

    if it goes from decreasing to increasing it is a local min.

OR plug and chug the **SIGN** of the **second derivative** of the all the points.

    Negative means concave down and it is a local max

    Positive means concave up and it is a local min

1. Find the absolute minimum of f(x) = $\frac{2x}{\sqrt{2x-3}}$ on the interval [2,6]

$f(x) = 2\left(\frac{x}{(2x-3)^{\frac{1}{2}}}\right)$

$f'(x) = 2\frac{\sqrt{2x-3}-x(\frac{1}{2})(\frac{1}{\sqrt{2x-3}})(2)}{2x-3}$ going to replace $(2x-3)^{\frac{1}{2}}$ *with* $\sqrt{2x-3}$ to make it look nicer

$f'(x) = 2\frac{\frac{\sqrt{2x-3}}{\sqrt{2x-3}}*(\sqrt{2x-3})-(\frac{x}{\sqrt{2x-3}})}{2x-3}$ multiply first term in numerator by $\frac{\sqrt{2x-3}}{\sqrt{2x-3}}$

$f'(x) = 2\frac{\frac{(2x-3)}{\sqrt{2x-3}}-(\frac{x}{\sqrt{2x-3}})}{2x-3}$

$f'(x) = 2\frac{(\frac{2x-3-x}{\sqrt{2x-3}})}{2x-3}$

$f'(x) = \frac{2(x-3)}{(2x-3)(\sqrt{2x-3})}$

Case 1

0 = $\frac{2x-6}{(2x-3)(\sqrt{2x-3})}$

$2x-6=0$

$x = 3$

Case 2 and 3

0 = $(2x-3)(\sqrt{2x-3})$

0 =2x-3 OR 0 = $(\sqrt{2x-3})$

$x=\frac{3}{2}$ OR $0 = 2x-3$ *then* $x=\frac{3}{2}$ This case is outside of our interval

So now we plug and chug $x=\frac{3}{2}$, $x = 3$, x = 2, and x=6

$f(3) = \frac{2(3)}{\sqrt{2(3)-3}}$ $f(3) = \frac{6}{\sqrt{3}}$	$f(2) = \frac{2(2)}{\sqrt{2(2)-3}}$ $f(2) = \frac{4}{\sqrt{1}}$ $f(2) = 4$	$f(2) = \frac{2(6)}{\sqrt{2(6)-3}}$ $f(2) = \frac{12}{\sqrt{9}}$ $f(2) = 4$

Now what is smaller $\frac{6}{\sqrt{3}}$ *or* 4?

$(\frac{6}{\sqrt{3}})^2$ square both of them = $\frac{36}{3}$ = 12	4^2=16

Therefore $\frac{6}{\sqrt{3}}$ is the absolute min. You can also estimate the root of 3 by hand to prove it.

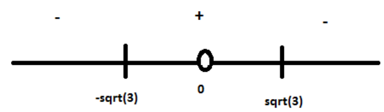
**** YOU DO NOT NEED TO KNOW THIS. Here's a little explanation to the logic of how my proof works. Let's assume we have positive numbers A and B and B>A. Therefore, B can be written as A+x where x is some positive number. If we are to square both we can write them as:

A^2 and $B^2 = (A+x)^2 = A^2 + 2Ax + x^2$

$A^2 < A^2 + 2Ax + x^2 = B^2$

As you can see if you were to square them the inequality will never change.

2. List out all the critical points of $f(x) = \frac{x^2-1}{x^3}$, list out the increasing and decreasing intervals, list out the local max and min



$f'(x) = \frac{(2x)(x^3)-(x^2-3)(3x^2)}{x^6}$

$f'(x) = \frac{2x^4-(3x^4-3x^2)}{x^6}$

$f'(x) = \frac{-x^4+3x^2}{x^6}$

$f'(x) = \frac{-x^2+3}{x^4}$

Case 1:

$0 = \frac{-x^2+3}{x^4}$

$0 = -x^2 + 3$

$x^2 = 3$

$x = \pm\sqrt{3}$

Case 2:

$0 = x^4$

$x = 0$

Test the sign of $f'(x)$ in the intervals

$f'(-2) < 0, \quad f'(1) > 0, f'(2) < 0$

Critical Points: $-\sqrt{3}, \sqrt{3}$

Increasing: $(-\infty, -\sqrt{3})$ *and* $(\sqrt{3}, \infty)$

Decreasing: $(-\sqrt{3}, 0)$ *and* $(0, \sqrt{3})$

Local min $(-\sqrt{3}, \frac{-2}{3\sqrt{3}})$

Local max $(\sqrt{3}, \frac{2}{3\sqrt{3}})$

V. CURVE SKETCHING

What to do:

1. Domain
2. Asymptotes
3. x and y intercept
4. Intervals of increase/decrease
5. Local max/min
6. Concavity
7. Draw!

1. Sketch the curve $f(x) = 3x^5 - 5x^3$

1. Domain: all real numbers

2. Asymptotes:

Vertical: none

Horizontal: take limit as x-> infinity and negative infinity and you will not get a finite number as a result. Therefore, no horizontal asymptotes.

3. Intercepts:

x intercept:

$0 = 3x^5 - 5x^3$

$0 = x^3(3x^2 - 5)$

x = 0, x = $\pm\sqrt{\frac{5}{3}}$

(0,0), $(-\sqrt{\frac{5}{3}}, 0)$, $(\sqrt{\frac{5}{3}}, 0)$

y intercept:

$y = 3(0^5) - 5(0^3)$

y = 0

(0,0)

4. Intervals of increase and decrease:

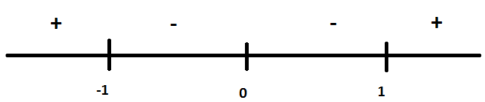
$f'(x) = 15x^4 - 15x^2$

$0 = 15x^2(x^2 - 1)$

x = 0 $\quad 0 = x^2 - 1$

$x^2 = 1$

$x = \pm 1$



$f'(-2) > 0$	$f'(-0.5) < 0$	$f'(0.5) < 0$	$f'(2) > 0$

Increasing on $(-\infty, -1)$ *and* $(1, \infty)$

Decreasing on $(-1,0)$ *and* $(0,1)$

5. Local max/min

At x = -1. $f(-1) = 3(-1)^3 - 5(-1)^3 = -3 - (-5) = 2$

Local max (-1,2)

At x=1. $f(1) = 3(1)^3 - 5(1)^3 = 3 - 5 = -2$

Local min (1,2)

6. Concavity

$f'(x) = 15x^4 - 15x^2$

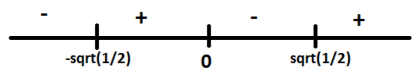
$f''(x) = 60x^3 - 30x$

$0 = 30x(2x^2 - 1)$

$x = 0 \qquad 0 = 2x^2 - 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$



$f(-1) < 0$	$f(-0.1) > 0$	$f(0.1) < 0$	$f(1) > 0$

Concave down on $\left(-\infty, -\sqrt{\frac{1}{2}}\right)$ *and* $\left(0, \sqrt{\frac{1}{2}}\right)$

Concave up on $\left(-\sqrt{\frac{1}{2}},\ 0\right)$ *and* $\left(\sqrt{\frac{1}{2}},\ \infty\right)$

7. sketch