MATH 104/184

2017W1 Midterm2 Review Package

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[24] **1. Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Find $\frac{dy}{dx}$ where $y = \log(\cot(\log(x)))$.

Answer:

(b) Find the derivative of $f(x) = (x^2 + 1)^4(x^3 - 12)^2\sqrt{x^{3/2} - 12}$.

Answer:

(c) Let $f(x) = \frac{\log x}{x^{2/3}}$. Find all critical points, global, local max/min, if any.

Answer:

(d) A spherical snowball is rolling down the hill and accumulating snow at the rate of 50 cm^3/min. How fast is the radius r increasing at the moment $r = 5$ cm?

Answer:

(e) The bookstore prints calculus textbooks on demand. They are currently selling 512 textbooks a year for \$64 each. The yearly demand equation is $q = 1024 - p^{1.5}$. Should they increase or decrease the price of the textbook to increase the revenue? What price would yield the maximum revenue?

Answer:

(f) Find the equation of the tangent line to the curve $x + \cos(x^2 y) - 2y = 3$ at the point $(2, 0)$.

Answer:

(g) Find the point in $(-\infty, 0)$ where tangent line to the equation $e^{2/x}$ has minimum slope.

Answer:

(h) When a company produces q notebooks, the average cost $C(q)/q$ is given by the function

$$y = \frac{20}{q} + \frac{16}{\sqrt{q}} - \frac{81}{q3^q}.$$

What is the approximate cost of the 100th notebook?

Answer:

Long Problems. In questions 2 - 6, show your work. No credit will be given for the answer without the correct accompanying work.

[8] **2.**

(a) You invest \$150,000 at an annual interest rate of 5%, compounded continuously. When the rate of growth of your investment hits \$10,000 a year, you plan to quit civilization and move to a tropical island. How long do you have to wait? [4pts]

(b) Unfortunately, you find out that your evil twin brother impersonated you and withdrew your money on the first day you invested it so he could gamble. Fortunately, in a remorseful moment, he re-deposits half of it in a bank that offers a simple annual interest rate of 8%. How long will it take for you to get back to the same amount you started with? [4pts]

[10] **3**.
The curve given by $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$, where a is a constant, is called a lemniscate, and looks something like the ∞ symbol or a sideways figure 8. The origin $(0, 0)$ is a singular point and has no well-defined tangent direction.

(a) Suppose $a = 2$. Find all the points on the lemniscate where the tangent line to the curve is horizontal and write down the equations of the lines. [5pts]

(b) Find the points where the normal line to the curve is horizontal. [5pts]

[12] **4**. True or False. Justify your answer or give a counterexample.

(a) If $f(0) = 2$ and $f(2) = 2$, then there exists some c in the interval $[0, 2]$ satisfying $f'(c) = 0$. [3pts]

(b) Any polynomial must have an absolute maximum on the interval $[-2, 2]$. [3pts]

(c) If f is differentiable everywhere and $f(x) = -1$ has two solutions, then $f'(x) = 0$ has at least one solution. [3pts]

(d) Let f be a differentiable function such that $f(1) = 2$ and $f'(x) \geq 3$ on $[1, 2]$. Then $f(2) \neq 4$. [3pts]

[10] **5**. I buy a secondhand car for $12,000 to commute to school. The car is estimated to depreciate continuously at 5% a year.

(a) What will be the value of my car after T years? [5pts]

(b) In 4 years' time, I need cash to pay for emergency surgery on my puppy who was hit by a drunk driver. I accept the first offer I get and sell the car for $6,000. Assuming the original depreciation estimate was correct, how much of a bargain did my buyer get? [5pts]

[10] **6**. The length l of a rectangle shadow is decreasing at a rate of 3 cm/sec while its width w is increasing at a rate of 2 cm/sec. At a given instant of time, t_0, the length of the shadow is exactly 10cm and its width is 4 cm.

(a) Find the rate of change of the area of the rectangle with respect to time at t_0. [3pts]

(b) Find the rate of change of the perimeter of the rectangle with respect to time at t_0. [3pts]

(c) Find the rate of change of the diagonal of the rectangle with respect to time at t_0. [4pts]

[12] **7**. The price p in dollars and demand q for copies of my review package are related by the following equation:

$$p^3q + 20q + \sqrt{q^2 - 75} = 285.$$

(a) Find the elasticity of demand for the package. [3pts]

(b) If the current demand is $q = 10$, will the revenue increase or decrease if the price is raised? [3pts]

(c) At the current demand of $q = 10$, if I decrease the price by 5%, what is the percentage change in the demand? [3pts]

(d) Suppose the price increases at a rate of \$0.25 a month. How fast does the demand change at the current demand of $q = 10$? [3pts]

[6] **8**. A small farm wants to produce some maple syrup. It costs them $C(q)$ dollars to produce q barrels, where

$$C(q) = 4q^{5/2} + 192.$$

The average cost of production is defined to be $C(q)/q$. How much maple syrup should the company produce in order to minimize the average cost of production per litre? Justify your answer.

[8] **8** A full conical tank of height 10 meters and top radius of 6 meters is drained into a cylindrical tank of height 8 meters and radius 3 meters. If the water of the conical tank drops at a constant rate of .5 meters a minute, at what rate does the water in the cylindrical tank rise when the water level of the conical tank is 3 meters?

2. Cheat Sheet

2.1. **Trig, exponential, log.**

- trig derivatives
- exponential and logarithmic functions
- growth/decay models

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\log x = \frac{1}{x}$$

2.2. **Chain Rule.** (p. 157)

- Key: Figure out which is the outside function and which is the inside function.
- Then: Derive the outside, and multiply by the inside derivative.
- Can iterate recipe to multiple nested functions.
- **Do not confuse with the product rule!**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

2.3. **Implicit Differentiation.** (p. 173)

- Use when you can't isolate y as a function of x.
- Take $\frac{d}{dx}$ of both sides of equation and use chain rule. A lot.
- Remember $\frac{d}{dx}x = 1$, $\frac{d}{dx}y = \frac{dy}{dx}$.
- Gather all y' terms on one side and solve for it.
- Use $\frac{dy}{dx}$ as the slope to tangent line as before, or (-1/slope) of normal line

2.4. **Logarithmic Differentiation.** (p. 172)

- Take log then differentiate. Then multiply by the original function.

$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)} \Longrightarrow f'(x) = f(x)\left(\frac{d}{dx}\log f(x)\right)$$

  Useful because log turns products into sums, quotients into differences, and powers into products.
- Used for relative rates of change (e.g. elasticity)

2.5. **Compound Interest.** (Handout)

A future value
P beginning value (principal)
r interest rate (as decimal or fraction)
n compounding periods per year
t years invested

- compound interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- continuously compounded interest $A = Pe^{rt}$

2.6. **Business terminology and notation.** (Handout)

- **quantity** q (number of widgets)
- **price** p (price per widget)
- **Revenue** R (money you receive when you sell q widgets for price p)
- **Cost** $C(q)$ (what it costs to make widgets)
  - **Fixed cost** F (costs that don't depend on the number of widgets)
  - **Variable cost** $V(q)$ (costs that depend on the number of widgets)
- **Break-Even Points** (q where costs equal revenue $C(q) = R(q)$)
- **Profit** P (financial gain; difference between amount earned R and spent C)
- **Demand** (equation that expresses the relation between quantity and price)
- **Marginal Cost** MC (additional cost to make another widget)
- **Marginal Revenue** MR (additional revenue for making another widget)
- **Marginal Profit** MP (additional profit for making another widget. Equivalently, difference between marginal revenue and marginal cost. Profit is maximized when $MP = 0$.)
- **Elasticity of Demand** ϵ (percent change in quantity demanded divided by percent change in price). $\epsilon < 0$ by law of demand

  If $|\epsilon| > 1$, widget is *price elastic* (wide choice of widgets), revenue decreasing at that price
  If $|\epsilon| < 1$, widget is *price inelastic* (widget is necessary staple), revenue increasing at that price
  If $|\epsilon| = 1$, widget is *price unit elastic.*

In math:

$$
\begin{aligned}
R &= pq \\
C(q) &= F + V(q) \\
P &= R - C \\
\epsilon &= \frac{p}{q}\frac{dq}{dp} \\
\frac{dR}{dp} &= q(1 + \epsilon)
\end{aligned}
$$

'Marginal' (`means take the derivative of that function with respect to` q.)

$$MC = \frac{dC}{dq}$$
$$MR = \frac{dR}{dq}$$
$$MP = \frac{dP}{dq} = MR - MC = \frac{dR}{dq} - \frac{dC}{dq}$$

Maximum profit when $MP = 0$, or equivalently, $MR = MC$.

2.7. **Theorems.**

- Key: Understand what the theorem means. In plain English.
- Memorize these theorems! Verbatim.
- Which theorem to use? *(What are we trying to show?)*
- Check hypothesis satisfied
- What does the conclusion of the theorem imply for my problem?

2.7.1. *Rolle's Theorem.* (Baby Mean Value) (Theorem 2.13.1)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
  - f differentiable on (a, b)
  - $f(a) = f(b)$.
- Then:
  - There is a c, with $a < c < b$, such that $f'(c) = 0$.

  This just means if f has the same value at its endpoints, it must have a horizontal tangent somewhere in between

2.7.2. *Mean Value Theorem.* (Theorem 2.13.4)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
  - f differentiable on (a, b)
- Then:
  - There is a c, with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

  This just means there's a tangent to f with the same slope as that of the secant line between the endpoints. I.e., there's some point where the instantaneous rate of change equals the average rate of change.

2.7.3. *Extreme Value Theorem.* (Theorem 3.5.11)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
- Then:
  - f attains a max and min on $[a, b]$.

2.8. **Related Rates.**

- Key: Organize your head
- Draw a picture.
- Label everything. *(Pick good names.)*
- What's constant? What's changing?
- What are we looking for? *(Differentiate what? With respect to what? At what specific instant?)*
- Write equations that relate {things you know} $\leftrightarrow$ {things you want to know}
- Differentiate relations using chain rule. Solve. *(Only plug in numbers for variables at the very end. Constants can use numbers from the beginning.)*

2.9. **Min/Max/Crit.**

- Absolute Min/Max (Defn 3.5.3)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq c \leq b$.
  - If $f(c) \geq f(x)$ for every x in $[a, b]$, then $f(c)$ is an absolute maximum.
  - If $f(c) \leq f(x)$ for every x in $[a, b]$, then $f(c)$ is an absolute minimum.
- Local Min/Max (Defn 3.5.3)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq e < c < f \leq b$.
  - If $f(c) \geq f(x)$ for every x in the open interval $(e, f) \subset [a, b]$, then $f(c)$ is a local maximum.
  - If $f(c) \leq f(x)$ for every x in the open interval $(e, f) \subset [a, b]$, then $f(c)$ is a local minimum.
- Critical Points (Defn 3.5.6)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq c \leq b$. If $f'(c)$ exists and is zero, we call $x = c$ a critical point.
- Check for possible global extreme values at (Theorem 3.5.12):
  - $f'(x) = 0$ (crit points)
  - $f'(x)$ DNE
  - endpoints of domain