MATH 104/184

FINAL EXAM REVIEW SESSION

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Introduction to Limits

The limit of a function is a value that the function **approaches** as the input **approaches** some value.

$$\lim_{x \to a} f(x) = c$$

This reads, "The limit of $f(x)$ as x approaches a is equal to c"

We say that the limit exists as x -> a when our function approaches the same value from both sides, that is, when:

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

Limits are important in differential calculus for a multitude of reasons: for the sake of this course, you'll need to know how to evaluate limit questions.

Your first step in evaluating a limit will be to plug in the value of a into your function.

1.) If you get a numerical value, or +-∞, then you can stop there.
2.) If you get an **indeterminate form**, then some manipulation is in order.

Examples of indeterminate forms: 0/0, ∞ / ∞, 0*∞, ∞ - ∞, 0^ ∞, ∞^ 0, 1^ ∞

Evaluate:

1.) $\lim_{x\to4}\left(\frac{x^2-16}{x^2-9x+20}\right)$

2.) $\lim_{x\to5}\left(\frac{x-5}{\sqrt{2x-6}-2}\right)$

3.) $\lim_{x\to-\infty}\left(\frac{\sqrt{9x^2-x+1}}{4x-5}\right)$

4.) $\lim_{x\to2}\left(\frac{x-2}{\left|\sqrt{x}-\sqrt{2}\right|}\right)$

Continuity & IVT

Continuity

A function f is continuous at x = a when:

1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$

In other words, the graph of the function is unbroken i.e. you could draw it from any point to another without lifting a pen from your paper.

It's important to note that many types of functions, such as polynomial functions, exponential functions, sqrt(x), log(x), sin(x) and cos(x) are continuous throughout their entire domains.

Intermediate Value Theorem

Theorem: If f is a continuous function whose domain is in the interval [a,b], then there exists at least one c within [a,b] such that f(a) < f(c) < f(b).

We have an interesting case of this theorem where f(a) is negative and f(b) is positive. By applying the IVT, we can demonstrate that there exists at least one c between a and b such that f(c) = 0. That is, f has at least one solution/root.

1.) Find the value of *a* for which the function f(x) is continuous for all x

$$f(x) = \begin{cases} 3x + a, x \leq e \\ 2a\ln x, x > e \end{cases}$$

2.) Prove that the following equation has a solution

$$2^x = x + e$$

Basics of Derivatives & L'Hopital's Rule

Derivatives

The derivative is the **instantaneous rate of change** of a function at a specific point.

The derivative of f(x) at x=a is given by:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Or equivalently,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

… provided that the limit exists. We say that f(x) is **differentiable** where this limit exists.

We can use the bottom case there, plugging in x=a to find the derivative of our function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

There's plenty of rules and shortcuts you can use to skip having to evaluate this limit every time: a list of formulas will be included at the bottom of this package.

A useful property that the derivative has is that its value at x = a is equal to the slope of the line that is tangent to f(x) at x = a.

$f'(a)$ = the slope of the line tangent to f(x) at x = a

Derivative Notation

$f'(x)$ = The derivative of f(x)

$\frac{d}{dx}f(x)\ or\ \frac{df(x)}{dx}$ = The derivative of f(x) with respect to x

The upper notation, called Lagrange notation is seen often when working with single-variable functions.

The one on the bottom, Leibniz's notation, is very useful when working with expressions without functions, or with multiple variables (like in related rates or optimization problems)

1.) Find $f'(x)$, where $f(x) = e^{12x}\cos(x)$

2.) Let $f(x) = \sqrt{x}$. Use the definition of the derivative to find $f'(4)$. No marks will be given for the use of any differentiation rules.

L'Hopital's Rule

L'Hopital's Rule can be either a big time saver or a necessity when evaluating some indeterminate limits, so it can be useful to know.

For functions f and g which are differentiable over an open interval I except possibly at a point c contained in I, if $\lim_{x \to c}(\frac{f(x)}{g(x)})$ is an indeterminate limit in the form 0/0 or ∞ / ∞, $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x \to c}(\frac{f'(x)}{g'(x)})$ exists, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

1. Evaluate $\lim_{x \to 0} \frac{\sin(x)}{-0.5x}$

Implicit Differentiation

Implicit differentiation is a trick you can use to compute derivatives when we have a complication formula for our relation. Suppose we were asked to find $\frac{dy}{dx}$ for this relationship at the point x=1

$$y = y^3 + xy + x^3$$

First, we plug in x = 1 into the equation and solve, which returns y = -1. Keep this in mind for the next step.

To differentiate this relationship, we're doing to take the derivative on both sides. Note that we're taking the derivative on both sides with respect to x. Since y is a function of x, we have to apply the chain rule on our y terms.

$$\frac{dy}{dx} = 3y^2 * \frac{dy}{dx} + y + x * \frac{dy}{dx} + 3x^2$$

Next, we're going to gather our $\frac{dy}{dx}$ terms on one side of the equation.

$$\frac{dy}{dx} - 3y^2 * \frac{dy}{dx} - x * \frac{dy}{dx} = y + 3x^2$$

$$\frac{dy}{dx}(1 - 3y^2 - x) = y + 3x^2$$

Now, isolate for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{y + 3x^2}{1 - 3y^2 - x}$$

Now, we plug in our values of x=1 and y=-1 into the function.

$$\frac{dy}{dx} = \frac{-1 + 3}{1 - 3 - 1} = -\frac{2}{3}$$

1. Find the equation of the tangent lines tangent to the graph of $x^2 + y^2 = 25$ at x = 2

Applications in Econ and Business

In business problems, you're typically given two things: a demand relationship (a relation between p (price) and q (quantity)) as well as a cost relationship that links a firm's cost C(q) with quantity produced. Your goal with these questions is typically to maximize profit, maximize revenues, or analyze pricing decisions using elasticity. You should be able to do all of these, given the demand and cost curves.

Demand

Demand is a (usually linear) relationship between the price of a product/service and the quantity demand of said product/service. On an exam, any demand relationship will be **negative**: any increase in price will be accompanied by a decrease in quantity demanded.

Example: Suppose that you are a player in the MMORPG "Treestory" and you are trying to make profit selling "Work Gloves", a popular in-game item. You notice that when you try to sell your Work Gloves for $16 each, your customer demand is 20 units. For every $2 decrease in unit price, the customer demand goes up by 10 units. Find the demand function linking p and q.

Revenue

Revenue can be calculated as follows: $R(q) = p * q$. It represents how much money is coming into the firm as a result of its sales.

Marginal Revenue is the amount of revenue that an additional unit q sold adds to your revenue.

$$MR(q) = \frac{dR}{dq}$$

Costs

Costs are the expenses associated with production. There're several kinds of costs that are useful to know about.

Fixed costs are costs that do not vary with production (that is, they are constant.)

Variable costs are costs that vary with production (that is, each additional unit produced will cause you to incur extra costs.)

Average cost is a measure of cost equal to total cost divided by the number of units produced.

Marginal cost is the derivative of Cost: it is the additional cost generated by an increase in q.

$$MC(q) = \frac{dC}{dq}$$

Example: "Appleson: We Hate Children Inc." is a textbook manufacturer that prints paper textbooks for students to purchase. Every month, they rent an industrial-grade printer at $50 a month to produce economics textbooks, which cost $70 in printing and paper expenses each to produce. What are their fixed, variable, average, marginal, and total costs?

Profit and break-even points

Profit is defined as Revenue – Cost.

A break-even point is where Revenue = Cost -> where profit is exactly zero.

Profit is maximized where the profit function's derivative is equal to zero, or where marginal revenue and marginal cost are equal.

Example: A company's cost and demand curves are given by

$$p + \sqrt{q} = 150 \ \ and\ C(q) = 2500 + 6q$$

Determine the selling price which would produce the most profit.

Price Elasticity of Demand

The Price Elasticity of Demand (ε) is a measurement of how responsive quantity demanded is to a change in price. It is given by:

$$\varepsilon = \frac{dq}{dp} * \frac{p}{q}$$

Where (p,q) is the point on the demand curve whose elasticity we're interested in.

When $|\varepsilon| > 1$, an increase in price will decrease revenue.

When $|\varepsilon| < 1$, an increase in price will increase revenue.

When $|\varepsilon| = 1$, revenue is maximized.

Example: Suppose the price and quantity demanded of a product are related as follows:

$$p = 20 - q$$

1.) Find the price elasticity of demand when p = 8
2.) To maximize revenues, should the company increase or decrease their price?

Mean Value Theorem

Let a and b be real numbers where a < b. And let f(x) be a function such that

- f(x) is continuous over the closed interval [a,b], and
- f(x) is differentiable over the open interval (a,b)

then there is a c within (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Which we can also express as:

$$f(b) = f(a) + f'(c)(b - a)$$

Note: **Rolle's Theorem** is a special case of the MVT where $f(b) = f(a)$. Rolle's Theorem states that assuming that the conditions for the MVT are met, while $f(b) = f(a)$, then There exists a c such that

$$f'(c) = 0$$

Example:

Let f(x) be a differentiable function so that

$f(1) = 10$ and $-1 \leq f'(x) \leq 2$ everywhere.

Obtain upper and lower bounds on f(5)

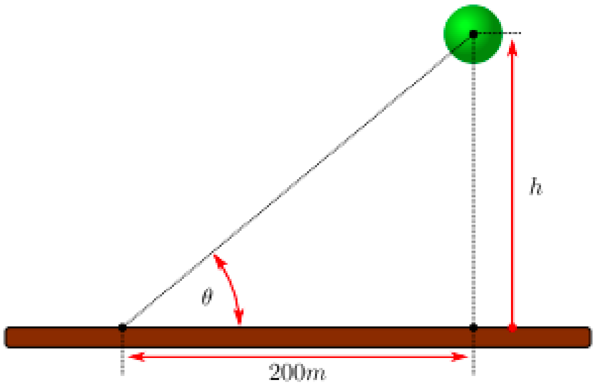
Related Rates

A related rates question is one where you are given the rate of change of a single variable and asked to find the rate of change of another.

<u>Consider the following question:</u>

Consider a helium balloon rising vertically from a fixed point 200m away from you. You observe that when the balloon is at an angle of $\frac{\pi}{4}$, its angle is changing at 0.05 radians per second. How fast is the balloon rising?

When looking at a question like this, it's always useful to draw a diagram. Here's a diagram from the Math textbook (Differential Calculus CLP1) that illustrates this scenario.



What other information do we have from the problem?

We are told that when $\theta = \frac{\pi}{4}, that\ \frac{d\theta}{dt} = 0.05$. We'll keep this in mind for later on.

Trigonometry tells us that:

$$tan\theta = \frac{h}{200} \qquad \rightarrow \qquad h = \ 200 * tan\theta$$

The REALLY BIG observation that you need to take away from this problem is that BOTH θ and h are functions of time.

When we differentiate both sides of the equation with respect to time, we therefore have to apply the chain rule (implicitly differentiate) on both the h and the θ.

$$\frac{dh}{dt} = 200 * sec^2\theta * \frac{d\theta}{dt}$$

Now, we can plug in the values of $\theta = \frac{\pi}{4}$ and $\frac{d\theta}{dt} = 0.05$ to solve.

$$\frac{dh}{dt} = 20m/s$$

If you know one related rates question, you know how to solve most of them. Here are some common steps you can take to solve most related rates problems.

1.) Read the question carefully. You'd be surprised at what kind of mistakes you can make if you don't fully read the question.
2.) Draw a diagram.
3.) Assign variables to the relevant quantities in the diagram.
4.) Find relations between the variables. Typically with questions involving geometric shapes, this will be the volume/surface area formula of said shape.
5.) Reduce the formula to a single variable. This is key – the calculus we know only allows us to work with one variable on the right hand side.
6.) Maximize or minimize
7.) Make sure your answer makes sense. You can't build a box with negative side lengths. You can't perform a task in negative time (unless you have a time machine)

1.) Water is being poured into a cone shaped cup at a rate of of $50cm^3$ per second. If the cup has a height of 8 cm and a top radius of 3cm, how fast is the water level rising when it is 4 cm full?

Optimization

One of the key goals in differential calculus is to find the maximum or minimum values of a function.

Global max/min

Let $a \leq b$ *and let* $f(x)$ *be defined over* $[a, b]$. *Now, let* $a \leq c \leq b$, *then*

$f(x)$ has a global maximum at $x = c$ if $f(x) \geq f(c)$ *for all* $a \leq c \leq b$

$f(x)$ has a global minimum at $x = c$ if $f(x) \leq f(c)$ *for all* $a \leq c \leq b$

WARNING: Notice that this includes the end points of the function.

Local max/min

Let $a \leq b$ *and let* $f(x)$ *be defined over* $[a, b]$. *Now, let* $a < c < b$, *then*

f(x) has a local maximum at x = c if there are a′ and b′ where $a \leq a' < c < b' \leq b$

Such that $f(x) \leq f(c)$ for all x where $a' < x < b'$

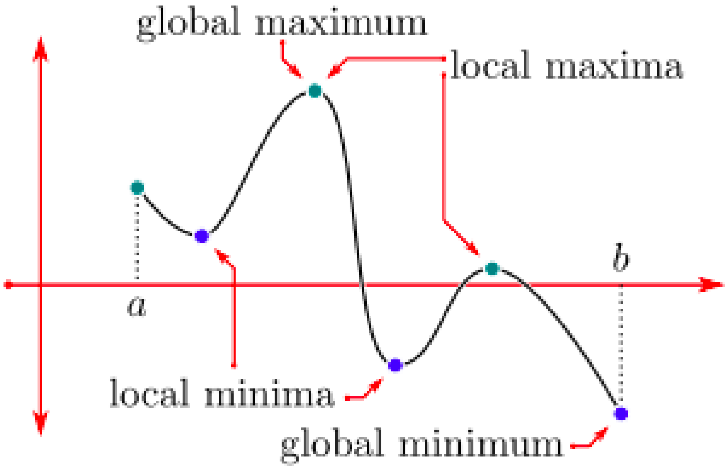
f(x) has a local minimum at x = c if there are a′ and b′ where $a \leq a' < c < b' \leq b$

Such that $f(x) \geq f(c)$ for all x where $a' < x < b'$

WARNING: This does not include the end points of the function.

Minimums and maximums are also referred to as “extrema” (i.e “the local extrema of a function)

Notice: if a function has a local extreme at $x = c$ and $f'(c)$ exists, $f'(c) = 0$



Source: (Differential Calculus CLP1)

Critical and Singular Points

If $f'(c)$ exists and is zero, we call $x = c$ a critical point of the function
If $f'(c)$ does not exist, we call $x = c$ a singular point of the function

When locating extrema, make sure you look at points where the $f'(x)$ is zero, and where it doesn't exist: more often than not, critical and singular points tend to be local or global max/mins.

1.) A closed rectangular container with a square base is to be made from two different materials. The material for the base costs \$5 per square meter, while the material for the other five sides costs \$1 per square meter. Find the dimensions of the container which has the largest possible volume if the total cost of materials is \$72.

2.) You are standing on the bank of a river that is 100m wide, and see 12 large kegs of beer calling your name 300m up the opposite shore. You can swim at 3m/s and run at 5m/s, and you want to get to the beer as quickly as possible. To what point on the opposite shore should you swim, before running the rest of the way?

Curve Sketching

Now that we've practiced our use of derivatives, we can move on to sketching the graphs of complicated functions.

Here are some good places to start when beginning to sketch a curve:

1.) Domain – Take note of values where x does not exist. This includes rational functions where the denominator can equal zero, square roots, and other sources of discontinuity
2.) Intercepts – plug in f(x) = 0 and x = 0 to find x and y intercepts.
3.) Vertical asymptotes – this applies to rational functions: values of x that make your function's denominator equal to zero have vertical asymptotes. Find the left and right hand side limits at these values to figure out how exactly your function is approaching the asymptotes.
4.) Horizontal asymptotes – evaluate the limits of your function as $x \to \pm\infty$. These values are your function's horizontal asymptotes. (When the limit evaluates to $\pm\infty$, no horizontal asymptote exists.

After you've completed these basic steps, **take the first derivative and find the intervals over which your function is increasing or decreasing.**

- If f'(x)>0 over (A,B), f(x) is increasing over (A,B)
- If f'(x)<0 over (A,B), f(x) is decreasing over (A,B)

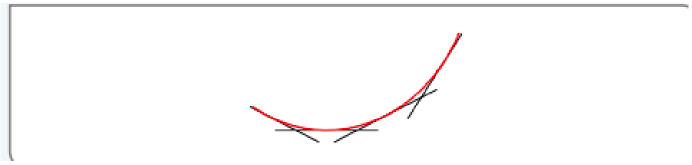
To find the intervals over which your function is increasing or decreasing, take the following steps:

- Find critical points. When you've found the points where $f'(x) = 0$, you can often use them as "barriers" to move forward (more on this in examples)
- Find places where the first derivative changes signs. For this to happen, our derivative has to pass through zero OR our function f must have a vertical asymptote.

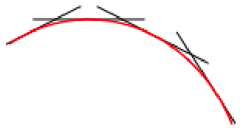
Now that that's done, **you have to examine your second derivative for concavity.**

Concavity

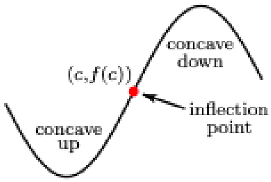
Where $f''(x) > 0, f(x)$ is said to be concave up.



Where $f''(x) < 0$, $f(x)$ is said to be concave down.



If at a point c, f''(x) changes sign, c is called an inflection point.



Now that we know the steps we need to take, we can start sketching some graphs.

1.) Sketch $f(x) = \frac{x}{x^2-4}$

2.) Sketch $f(x) = x^3 - 6x^2 + 9x - 54$

Taylor Polynomials

Suppose that you're interested in the values of a function f(x) for x near a certain point a. We can use polynomials that are easily calculable to approximate the value of our function near a certain point.

To perform a linear approximation at x near a:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Example: estimate $e^{0.1}$

For most curvy functions, we can do better than a linear approximation (if we're looking for accuracy).

To perform a quadratic approximation at x near a:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

For a third-degree polynomial, take this current formula and add a $+\frac{1}{6}f'''(a)(x-a)^3$ at the end of the quadratic.

For an nth degree Taylor Polynomial, take the (n-1)th degree polynomial and add the following term at the end

$$\frac{1}{n!}f^{(n)}(a)(x-a)^n$$

1) Compute the Taylor polynomial of degree 3 of $f(x) = xlnx$ at $a = 1$

2) Estimate $|\sin(0.12) - 0.12|$ by using the linear approximation of $sinx$ at $a = 0$

Appendix: Useful formulas

Note: this appendix only includes useful formulas that weren't previously mentioned.

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$